Develop an Inventory Model with Time Dependent Deterioration, and Demand Rate under the Condition of Permissible Delay

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ABSTRACT:
In this paper, an inventory model is considered with variable demand rate and time dependent deterioration. Here, we have taken shortages which are fully backlogged. This model is studied under the condition of permissible delay in payments which is very important and necessary aspect for any business to smooth running. The optimal replenishment policy for the model is derived. At last, a numerical example is given.

Keywords: Inventory, shortages, time dependent deterioration, exponential demand rate and trade credit

INTRODUCTION:
In the classical inventory model the deterioration is assumed to be a constant. Many researchers considered linearly time dependent deterioration, but in reality deterioration cannot be constant also it does not depend up on time only. Deterioration may be affected by storage condition, weather condition etc. The effect of deterioration of physical goods in stock is very realistic feature of inventory control. **Ghare** and **Schrader (1963)** were the first to develop an EOQ model for an item with exponential decay and constant demand. For more details about deteriorating items one can see the review paper of **Goyal** and **Giri (2001).**

In developing the models for different inventory systems, most of the researchers assumed that the retailer pays the amount to the supplier immediately as soon as the items are received. But in practice, this is not always true. The suppliers often offer their customers a fixed period of delay in payment. This grace period allows the customer to settle the account for payment of the amount owed to the supplier without charged interest. This event is known as trade credit policy. As a matter of fact, the permissible delay in payments gives a benefit to the supplier because this policy attracts to the new customers for increase the sales. **Goyal, S.K. (1985)** was the first to establish mathematical models with permissible delay in payment to determine the optimal order quantities. Afterwards, several studies were proposed to improve **Goyal's (1985)** model. **Aggarwal, S.P.** and **Jaggi, C.K. (1995)** extended **Goyal's (1985)** model to determine the order level model for deteriorating items under permissible delay in payments. The demand and deterioration were taken as constant and shortages were not allowed in their study. **Singh et al. (2007)** developed perishable inventory model with quadratic demand with partial backlogging. In this paper permissible delay in payment is also taken. **Teng, et al. (2007)** developed **r**etailer's optimal ordering policies with trade credit financing. Optimal replenishment policy for the EPQ Model with permissible delay in payments is considered by **Fei et al. (2010).** In this paper shortages are allowed. **Singh et al. (2009)** developed a production inventory model with variable demand rate for deteriorating items under permissible delay in payments. **Kumar et al. (2008), Teng, J.T., (2002). Yang P.C., Wee H.M., (2006), Jamal, M. M., B.R.Sarkar and S.Wang,(1997),** also considered the permissible delay in their papers. An optimal inventory policy for items having linear demand

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and variable deterioration under the condition of trade credit has discussed by **Singhet al. (2010)**.**Shah et al. (2011)** analyzed an inventory model with pricing policy and sensitive stock dependent demand under the progressive payment scheme. **Benkherouf (1995)** discussed an inventory model with deterioration and decreasing time dependent demand with shortages. **Chang, et al.2001)** developed an inventory model for deteriorating items with linear trend demand under the condition of permissible delay in payment. **Teng et al. (2002)** has discussed inventory model for deteriorating items with time varying demand and partially backlogging.

In this paper we have taken time dependent deterioration, and demand rate under the condition of permissible delay. The objective of the purposed model is to determine the optimal payment and replenishment cycle for a purchaser to minimize the cost.

ASSUMPTIONS AND NOTATIONS:
To develop the model, the following assumptions and notations are being made:

Assumptions
- The replenishment occurs instantaneously at an infinite rate.
- Shortages are allowed and fully backlogged.
- Lead time is zero.
- Deterioration is time dependent.
- Demand is taken as a exponential function of t

Notations
A	ordering cost per order
θ	Deterioration rate; taken as $\theta(t) = \theta t$, where $0 < \theta << 1$
D(t)	demand rate is a function of t such that $D(t) = ae^{bt}$,where a > 0 and b > 0.
I(t)	inventory level at time t
P	purchasing cost, \$/per unit,
C	selling price \$/per unit
H	holding cost excluding interest charges, \$/unit/year,
Cd	deterioration cost \$/per unit
Cs	shortage cost \$/per unit
Ie	interest which can be earned, \$/year,
Ir	interest charges which are invested in inventory, \$/year, Ir > Ie,
t1	the time at which the inventory level reaches zero; $t_1 \geq 0$;
M	permissible delay in settling the accounts, year
T	the length of replenishment cycle, year,
C(t1,T)	the total reverent inventory cost,
C1(t1,T)	the total reverent inventory cost for T>M in case I
C2(t1,T)	the total reverent inventory cost for $T \leq M$ in case II
V(t1,T)	the total reverent inventory cost,
V1(t1,T)	the total reverent inventory cost for T>M in case I
V2(t1,T)	the total reverent inventory cost for $T \leq M$ in case II

FORMULATION OF MATHEMATICAL MODEL
In the formulation of inventory model with above described assumption and notation is depicted in fig 1.During the time interval [0,t1] the depletion of inventory occurs due to combine effect of demand and deterioration. At

time t1 inventory level goes to zero and shortage occurs, which are fully backlogged. During the period (0, T), the variation of inventory level I(t), with respect of time can be described by the following differential equations:
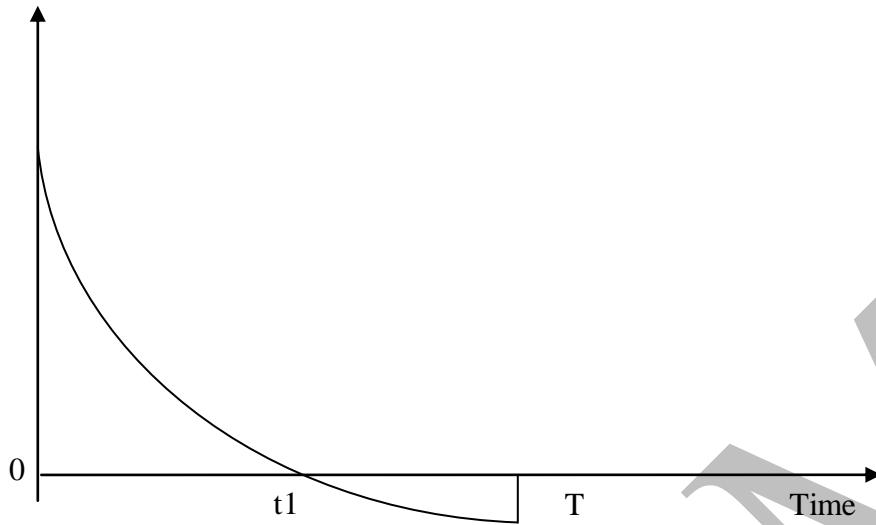
$$\frac{dI(t)}{dt} + \theta t = -ae^{bt}; \qquad 0 \le t \le t_1 \qquad\qquad (1)$$

$$\frac{dI(t)}{dt} = -ae^{bt}; \qquad t_1 \le t \le T \qquad\qquad (2)$$

With boundary conditions

$$I(0) = I_{max} \qquad\qquad (3)$$

$$I(T) = I_{min} \qquad\qquad (4)$$

$$I(t_1) = 0 \qquad\qquad (5)$$

The solutions of equations (1) and (2), using (5) are given by

$$I(t) = \frac{a}{b}\left((1+\frac{\theta}{b^2})(e^{bt_1} - e^{bt}) - \frac{\theta}{b}(t_1 e^{bt_1} - te^{bt}) + \frac{\theta}{2}(t_1^2 - t^2)e^{bt_1}\right); \qquad 0 \le t \le t_1 \qquad (6)$$

And

$$I(t) = \frac{a}{b}(e^{bt_1} - e^{bt}) \qquad\qquad t_1 \le t \le T \qquad\qquad (7)$$

At, t=0, I (0) = I$_{max}$ in equation (6), one have,

$$I_{max} = \frac{a}{b}\left((1+\frac{\theta}{b^2}) - \frac{\theta}{b}t_1 + \frac{\theta}{2}t_1^2\right)e^{bt_1} \qquad\qquad (8)$$

At, t=T, I (T) is minimum, therefore

$$I_{min} = \frac{a}{b}(e^{bt_1} - e^{bT}) \tag{9}$$

Therefore, the total ordering quantity

$$Q = I_{max} + I_{min}$$

$$Q = \frac{a}{b}((\frac{\theta}{b^2} - \frac{\theta}{b}t_1 + \frac{\theta}{2}t_1^2)e^{bt_1} - e^{bT}) \tag{10}$$

Now, we find all the related costs:

a) Cost of placing orders=A
b) Cost of purchasing = CQ

$$= C\frac{a}{b}((\frac{\theta}{b^2} - \frac{\theta}{b}t_1 + \frac{\theta}{2}t_1^2)e^{bt_1} - e^{bT}) \tag{11}$$

c) Cost of holding inventory=$\int_0^{t_1} I(t)dt$

$$= h\frac{a}{b}\{((1+\frac{\theta}{b^2})(t_1 - \frac{1}{b}) - \frac{\theta}{b}(t_1^2 - \frac{t_1}{b} + \frac{1}{b^2}) + \frac{\theta}{3}t_1^3)e^{bt_1} + \frac{1}{b} + 2\frac{\theta}{b^3}\} \tag{12}$$

d) Cost of shortages=$C_S[\int_{t_1}^{T}(-I(t))dt]$

$$= -C_s\frac{a}{b}((T - t_1 + \frac{1}{b})e^{bt_1} - \frac{1}{b}e^{bT}) \tag{13}$$

e) Cost of deterioration= $C_d\int_0^{t_1}(\theta t)I(t)dt$

$$= C_d\theta((\frac{t_1^2}{2} - \frac{t_1}{b} + \frac{1}{b^2})e^{bt_1} - \frac{1}{b^2}) \tag{14}$$

There are following two possible cases based on the values of t_1 and m.

- **Case I:** $t_1 \geq M$ (Payment at or before total depletion of inventory i.e. the inventory not being sold after the due date and evaluate the interest payable IP1 and interest earned IE1 per cycle)
- **Case II:** $t_1 < M$. (Payment at or after depletion i.e. the interest payable per cycle is zero because the supplier can be paid in full at time M , So only evaluate the interest earned per cycle Which is earned during the positive inventory period plus the interest earned from the cash invested during time period (t1 , M) after the inventory is exhausted at time T.

Case I: $t_1 \geq M$

In this case, the credit time expires on or before the inventory depleted completely to zero. The interest payable per cycle for the inventory not being sold after the due date M is Interest payable in the time horizon When M < t ≤ t1

$$\text{Interest Earned} \qquad = PI_e\int_0^{M} tae^{bt}dt$$

$$= aPI_e(\frac{M}{b}e^{bM} - \frac{1}{b^2}e^{bM} + \frac{1}{b^2}) \qquad (15)$$

Interest payable $\qquad = CI_r \int_M^{t_1} I(t)dt$

$$= CI_r(((1+\frac{\theta}{b^2})(t_1 - \frac{1}{b} - M) - \frac{\theta}{b}(t_1^2 - t_1(M+\frac{1}{b}) + \frac{1}{b^2}) + \frac{\theta}{2}(2t_1^3 - Mt_1^2 + \frac{M^3}{3})e^{bt_1}$$

$$+(1 + 2\frac{\theta}{b^2} - \frac{\theta M}{b})\frac{e^{bM}}{b})$$

$$\qquad (16)$$

Case II: $t_1 < M$.

In this case, the interest payable per cycle is zero, i.e., $IP2 = 0$, when $t_1 < M \le T$ because the supplier can be paid in full at time M , the permissible delay. Thus, the interest earned per cycle is the interest earned during the positive inventory period plus the interest earned from the cash invested during time period (t1 , M) after the inventory is exhausted at time t1 , and it is given by

Interest Earned $\qquad = PI_e(\int_0^{t_1} ae^{bt}tdt + (M - t_1)\int_0^{t_1} ae^{bt}dt)$

$$= PI_e((\frac{1}{b^2} - \frac{M}{b})(1 - e^{bt_1}) + \frac{t_1}{b}) \qquad (18)$$

Total cost is given by

$C(t_1, T) = $ (Cost of placing orders + Cost of purchasing + Cost of holding inventory + Cost of deterioration + Cost of shortages + Interest payable per year - Interest earned per year)

Total Annual Cost

$$V(t_1, T) = \frac{1}{T}C(t_1, T)$$

Case I

$$C_1(t_1, T) \quad = \quad A + C\frac{a}{b}((\frac{\theta}{b^2} - \frac{\theta}{b}t_1 + \frac{\theta}{2}t_1^2)e^{bt_1} - e^{bT}) +$$

$$h\frac{a}{b}\{((1+\frac{\theta}{b^2})(t_1 - \frac{1}{b}) - \frac{\theta}{b}(t_1^2 - \frac{t_1}{b} + \frac{1}{b^2}) + \frac{\theta}{3}t_1^3)e^{bt_1} + \frac{1}{b} + 2\frac{\theta}{b^3}\} \quad -C_s\frac{a}{b}((T - t_1 + \frac{1}{b})e^{bt_1} - \frac{1}{b}e^{bT}) +$$

$$C_d\theta((\frac{t_1^2}{2} - \frac{t_1}{b} + \frac{1}{b^2})e^{bt_1} - \frac{1}{b^2}) + \quad CI_r(((1+\frac{\theta}{b^2})(t_1 - \frac{1}{b} - M) - \frac{\theta}{b}(t_1^2 - t_1(M+\frac{1}{b}) + \frac{1}{b^2}) + \frac{\theta}{2}(2t_1^3 - Mt_1^2 + \frac{M^3}{3})e^{bt_1}$$

$$+(1 + 2\frac{\theta}{b^2} - \frac{\theta M}{b})\frac{e^{bM}}{b}) - aPI_e(\frac{M}{b}e^{bM} - \frac{1}{b^2}e^{bM} + \frac{1}{b^2})$$

$$V_1(t_1,T) = \frac{1}{T}[\ A + C\frac{a}{b}((\frac{\theta}{b^2} - \frac{\theta}{b}t_1 + \frac{\theta}{2}t_1^2)e^{bt_1} - e^{bT}) +$$

$$h\frac{a}{b}\{((1+\frac{\theta}{b^2})(t_1 - \frac{1}{b}) - \frac{\theta}{b}(t_1^2 - \frac{t_1}{b} + \frac{1}{b^2}) + \frac{\theta}{3}t_1^3)e^{bt_1} + \frac{1}{b} + 2\frac{\theta}{b^3}\} \quad -C_s\frac{a}{b}((T - t_1 + \frac{1}{b})e^{bt_1} - \frac{1}{b}e^{bT}) +$$

$$C_d\theta((\frac{t_1^2}{2} - \frac{t_1}{b} + \frac{1}{b^2})e^{bt_1} - \frac{1}{b^2}) + \quad CI_r(((1+\frac{\theta}{b^2})(t_1 - \frac{1}{b} - M) - \frac{\theta}{b}(t_1^2 - t_1(M+\frac{1}{b}) + \frac{1}{b^2}) + \frac{\theta}{2}(2t_1^3 - Mt_1^2 + \frac{M^3}{3})e^{bt_1}$$

$$+(1+2\frac{\theta}{b^2} - \frac{\theta M}{b})\frac{e^{bM}}{b}) - aPI_e(\frac{M}{b}e^{bM} - \frac{1}{b^2}e^{bM} + \frac{1}{b^2})\ ] \qquad (19)$$

The total annual cost has the two variables t_1 and T. To minimize the total annual cost, the optimal values of t_1 and T can be obtained by solving the following equations simultaneously

$$\frac{\partial V_1(t_1,T)}{\partial t_1} = 0 \qquad (20)$$

and

$$\frac{\partial V_1(t_1,T)}{\partial T} = 0 \qquad (21)$$

Provided, they satisfy the following conditions

$$\frac{\partial^2 V_1(t_1,T)}{\partial t_1^2} > 0, \frac{\partial^2 V_1(t_1,T)}{\partial T^2} > 0$$

and

$$\left(\frac{\partial^2 V_1(t_1,V)}{\partial t_1^2}\right)\left(\frac{\partial^2 V_1(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 V_1(t_1,T)}{\partial t_1 \partial T}\right)^2 < 0 \qquad (22)$$

Case II

$$C_2(t_1,T) = A + C\frac{a}{b}((\frac{\theta}{b^2} - \frac{\theta}{b}t_1 + \frac{\theta}{2}t_1^2)e^{bt_1} - e^{bT})$$

$$+h\frac{a}{b}\{((1+\frac{\theta}{b^2})(t_1 - \frac{1}{b}) - \frac{\theta}{b}(t_1^2 - \frac{t_1}{b} + \frac{1}{b^2}) + \frac{\theta}{3}t_1^3)e^{bt_1} + \frac{1}{b} + 2\frac{\theta}{b^3}\} -C_s\frac{a}{b}((T - t_1 + \frac{1}{b})e^{bt_1} - \frac{1}{b}e^{bT}) +$$

$$C_d\theta((\frac{t_1^2}{2} - \frac{t_1}{b} + \frac{1}{b^2})e^{bt_1} - \frac{1}{b^2}) - PI_e((\frac{1}{b^2} - \frac{M}{b})(1 - e^{bt_1}) + \frac{t_1}{b})$$

$$V_2(t_1,T) = \frac{1}{T}[\ A + C\frac{a}{b}((\frac{\theta}{b^2} - \frac{\theta}{b}t_1 + \frac{\theta}{2}t_1^2)e^{bt_1} - e^{bT})$$

$$+h\frac{a}{b}\{((1+\frac{\theta}{b^2})(t_1 - \frac{1}{b}) - \frac{\theta}{b}(t_1^2 - \frac{t_1}{b} + \frac{1}{b^2}) + \frac{\theta}{3}t_1^3)e^{bt_1} + \frac{1}{b} + 2\frac{\theta}{b^3}\} -C_s\frac{a}{b}((T - t_1 + \frac{1}{b})e^{bt_1} - \frac{1}{b}e^{bT}) +$$

$$C_d\theta((\frac{t_1^2}{2} - \frac{t_1}{b} + \frac{1}{b^2})e^{bt_1} - \frac{1}{b^2}) - \qquad PI_e((\frac{1}{b^2} - \frac{M}{b})(1 - e^{bt_1}) + \frac{t_1}{b})\ ]$$

$$(23)$$

The total annual cost has the two variables t_1 and T. To minimize the total annual cost, the optimal values of t_1 and T can be obtained by solving the following equations simultaneously

$$\frac{\partial V_2(t_1,T)}{\partial t_1} = 0 \tag{24}$$

and

$$\frac{\partial V_2(t_1,T)}{\partial T} = 0 \tag{25}$$

Provided, they satisfy the following conditions

$$\left. \begin{array}{l} \dfrac{\partial^2 V_2(t_1,T)}{\partial t_1^2} > 0, \dfrac{\partial^2 V_2(t_1,T)}{\partial T^2} > 0 \\[2em] \left(\dfrac{\partial^2 V_2(t_1,V)}{\partial t_1^2} \right)\left(\dfrac{\partial^2 V_2(t_1,T)}{\partial T^2} \right) - \left(\dfrac{\partial^2 V_2(t_1,T)}{\partial t_1 \partial T} \right)^2 < 0 \end{array} \right\} \tag{26}$$

and

The equations (19) and (23) are solved for the optimum value of cost. To minimize the objective function, the optimal solution of t_1 and T can be obtained from the equation (20), (21), (24) and (25) for both the cases. All these equations are solved numerically with the help of Computer Software.
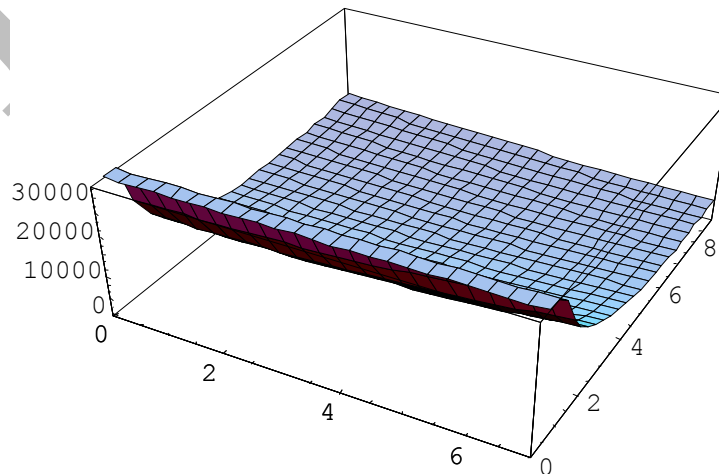
NUMERICAL ILLUSTRATIONS
The applications of the above theoretical results are illustrated by the numerical example. The following data in appropriate units based on the previous studies has been utilized:
A=350   a=200   b=0.08   P=18,   Ch=0.5   Ie=0.03   .Ir=0.09 θ=0.04   Cs=0.5   C=15
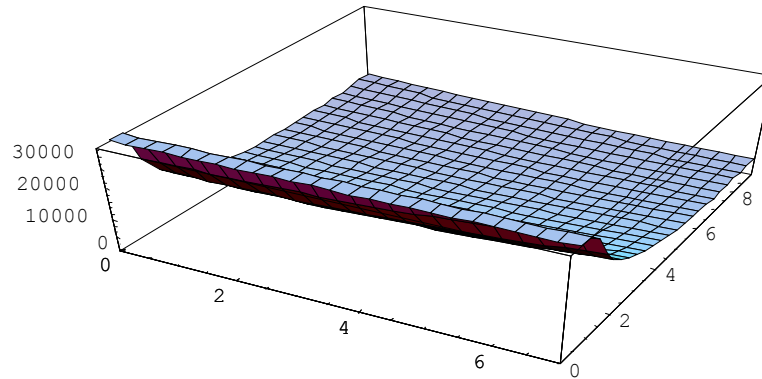Cd=0.05

Case I: M=5
We get   t1=6.00484,   T=9.32359,   and   V1 = 5040.24

Convexity of the Total Cost Function for case I

CASE II: M=10
We get   t1=9.17561,   T=11.4885,   V2 = 5465.34

Convexity of the Total Cost Function for case II

CONCLUSION:

In this paper, we developed an order level inventory model for deteriorating items. In this paper, we develop an inventory model with time dependent deterioration, which is a realistic situation for many items during storage period. Here, we have taken demand rate is an exponential increasing function of time, which shows the small change in time, demand is increasing a lot, finally we have taken permissible delay which is very real situation now a days. In real life situations, the supplier frequently offers a permissible delay to attract the customer for increasing their ordering quantity. Finally we have given a numerical example to justify our model. The proposed model can be extended in several ways, for instance we may consider finite planning horizon. Also we can extend the deterministic demand function to stochastic demand patterns. Furthermore, we can generalize the model to allow permissible delay in payments.

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